Integration By Parts
Part II - Weighted Average Revenue Life

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Integration by parts is a method of integration that is often useful when two functions are multiplied together. In Part I we developed the mathematics of integration by parts and used the mathematics to solve a simple hypothetical problem. In Part II of the series we will use integration by parts to determine the weighted average life of a revenue stream that declines over time. To that end we will use the following hypothetical problem...

Our Hypothetical Problem

Products in the high-tech space evolve to accommodate competitive market pressures, rapid rates of technology change, and constant improvements in performance and functionality. While adding functionality and value, the fast moving technologies also make products obsolete quickly. Technological obsolescence usually occurs when a new product or technology supersedes the old, and it becomes preferred to use the new technology in place of the old, even if the old product is still functional.

We will define the variable R_t to be annualized revenue at time t and the variable λ to be the rate of revenue decline (this rate is constant over time). Assume that we have the following go-forward annualized revenue equation for a tech company whose sole product is subject to technological obsolescence...

$$R_t = R_0 \,\mathrm{Exp}\left\{-\lambda\, t\right\} \text{ ...where... } R_0 = 10{,}000{,}000 \text{ ...where... } \lambda = 0.30 \tag{1}$$

Our job is to estimate the weighted average life (in years) of the above product revenue stream. We want to answer the following questions...

Question 1: What is the weighted average life of the above revenue stream over the time interval $[0, \infty]$?

Question 2: Given that we want a weighted average life of four years what is the value of λ?

The Mathematics

We will define the variable WAL to be the weighted average life (in years) of a revenue stream. Weighted average life is defined as time-weighted cumulative revenue divided by unweighted cumulative revenue. Using Equation (1) above the equation for the weighted average life of revenue received over the time interval $[0, \infty]$ is...

$$WAL = \int_0^\infty u\, R_0 \,\mathrm{Exp}\left\{-\lambda\, u\right\} \delta u \div \int_0^\infty R_0 \,\mathrm{Exp}\left\{-\lambda\, u\right\} \delta u \tag{2}$$

The first integral in Equation (2) above is the integral of the product of two functions and therefore we will solve that integral via integration by parts. The integral that we want to solve is...

$$I_1 = \int_0^\infty u \,\mathrm{Exp}\left\{-\lambda\, u\right\} \delta u \tag{3}$$

We will make the following definitions...

$$v(u) = u \text{ ...where... } \frac{\delta v(u)}{\delta u} = 1 \text{ ...and... } w(u) = -\frac{1}{\lambda}\,\mathrm{Exp}\left\{-\lambda\, u\right\} \text{ ...where... } \frac{\delta w(u)}{\delta u} = \mathrm{Exp}\left\{-\lambda\, u\right\} \tag{4}$$

Using the definitions in Equation (4) above we can rewrite Equation (3) above as...

$$I_1=\int\limits_0^\infty v(u)\,\frac{\delta w(u)}{\delta u}\,\delta u \tag{5}$$

Using integration by parts we can rewrite Equation (5) above as... [1]

$$I_1=v(\infty)\,w(\infty)-v(0)\,w(0)-\int\limits_0^\infty w(u)\,\frac{\delta v(u)}{\delta u}\,\delta u \tag{6}$$

Using the definitions in Equation (4) above and Appendix Equation (16) below we can rewrite Equation (6) above as...

$$\begin{aligned}I_1&=-\frac{1}{\lambda}\,(b-a)\,\mathrm{Exp}\Big\{-\lambda\,b\Big\}+\frac{1}{\lambda}\,(a-a)\,\mathrm{Exp}\Big\{-\lambda\,a\Big\}+\frac{1}{\lambda}\int\limits_a^b\mathrm{Exp}\Big\{-\lambda\,u\Big\}\,\delta u\\&=-\frac{1}{\lambda}\,(b-a)\,\mathrm{Exp}\Big\{-\lambda\,b\Big\}+\frac{1}{\lambda^2}\left(\mathrm{Exp}\Big\{-\lambda\,a\Big\}-\mathrm{Exp}\Big\{-\lambda\,b\Big\}\right)\end{aligned} \tag{7}$$

Using Appendix Equation (16) below the solution to the second integral in Equation (**??**) above is...

$$I_2=\int\limits_a^b\mathrm{Exp}\Big\{-\lambda\,u\Big\}\,\delta u=\frac{1}{\lambda}\left(\mathrm{Exp}\Big\{-\lambda\,a\Big\}-\mathrm{Exp}\Big\{-\lambda\,b\Big\}\right) \tag{8}$$

Using Equations (7) and (8) above we can rewrite weighted average life Equation (**??**) above as...

$$WAL=I_1\div I_2 \tag{9}$$

The Solution To Our Hypothetical Problem

Question 1: What is the weighted average life of the above revenue stream over the time interval $[0,\infty]$?

The bounds of integration are...

$$a=0\ \text{...and...}\ b=\infty \tag{10}$$

Using Equation (7) above the value of I_1 is...

$$I_1=-\frac{1}{\lambda}\,(\infty-0)\,\mathrm{Exp}\Big\{-\lambda\times\infty\Big\}+\frac{1}{\lambda^2}\left(\mathrm{Exp}\Big\{-\lambda\times 0\Big\}-\mathrm{Exp}\Big\{-\lambda\times\infty\Big\}\right)=\frac{1}{\lambda^2} \tag{11}$$

Using Equation (8) above the value of I_2 is...

$$I_2=\frac{1}{\lambda}\left(\mathrm{Exp}\Big\{-\lambda\times 0\Big\}-\mathrm{Exp}\Big\{-\lambda\times\infty\Big\}\right)=\frac{1}{\lambda} \tag{12}$$

Using Equations (9), (11) and (12) above weighted average life is...

$$WAL=\frac{I_1}{I_2}=\frac{\frac{1}{\lambda^2}}{\frac{1}{\lambda}}=\frac{1}{\lambda} \tag{13}$$

For our hypothetical problem using Equation (13) above weighted average life is...

$$WAL=\frac{1}{0.30}=3.33\,\text{years} \tag{14}$$

Question 2: Given that we want a weighted average life of four years what is the value of λ?

Using Equation (13) above the value of λ is...

$$\text{if...}\ WAL=\frac{1}{\lambda}=4\ \text{...then...}\ \lambda=\frac{1}{WAL}=\frac{1}{4}=0.25 \tag{15}$$

References

[1] Gary Schurman, *Integration By Parts - Part I,* January, 2020

Appendix

A. The solution to the following integral is...

$$\int\limits_a^b \mathrm{Exp}\left\{-\lambda\,u\right\}\delta u = -\frac{1}{\lambda}\,\mathrm{Exp}\left\{-\lambda\,u\right\}\Big[_a^b$$
$$= -\frac{1}{\lambda}\left(\mathrm{Exp}\left\{-\lambda\,b\right\} - \mathrm{Exp}\left\{-\lambda\,a\right\}\right)$$
$$= \frac{1}{\lambda}\left(\mathrm{Exp}\left\{-\lambda\,a\right\} - \mathrm{Exp}\left\{-\lambda\,b\right\}\right) \tag{16}$$